Technical Report No. 32-655
(Part III)

Mathematical Models of Missile Launching

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ABSTRACT

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By means of a mathematical model of the situation, including assumptions about the nature of the delays encountered, an estimate is presented of the expected number of days required to launch four missiles from two pads, allowing no simultaneous (i.e., same-day) countdowns or launches.

Author

I. INTRODUCTION

In space mission feasibility studies it is often necessary to have an estimate of the number of days required to launch a given number of missiles from another given (and possibly different) number of pads. This report (Part III of three parts) is one of a series whose purpose is to investigate this question with the aid of probability theory, for different launching configurations of interest (see Ref. 1–3). The particular launching configuration examined in this report is that one of four missiles being launched from two pads in N days, allowing no simultaneous (i.e., same-day) countdowns or launches. The following assumptions constitute the launching model.

II. THE MATHEMATICAL LAUNCH MODEL

1. There are four identical missiles.
2. For a missile erected and counting, a complete countdown and launching in one day is possible, though not necessary. But once the counting has started on any given missile, the counting on that missile continues until that missile fires, no matter how many days this requires. There is a probability p (a specified constant) that a missile will successfully count down and fire in a single day. There is thus also a constant probability $q = 1 - p$ that the missile will fail to complete a countdown on that day, and will incur a one-day delay. In other words, if the countdown on a given missile stops on any day of counting, counting down on that missile must start over from the beginning, the following morning.
3. Because p is a constant, failures to fire a given missile do not influence the value of p in subsequent attempts to fire the same missile or others. Each missile is thus probabilistically independent of the other three.
4. There is for both pads the same turnaround time R days (a specified constant), starting on the day

after a launching, during which both the pad is cleaned up and a new missile is erected. The new missile is thus ready to start counting on the $(R+1)$th day (but it does not necessarily have to start counting).

5. Only one of the missiles can be counting down on any given day, and after a given missile fires, the next missile to be fired must wait until the next day before starting to count down, if it is ready to count.

6. Prior to the start of the N-day period, one missile is erected on each of the two pads, ready to start counting. On the first day of the N-day period, counting down is initiated on one of the missiles, and continues until that missile fires. From here four separate sequences of launches are possible, as will be discussed in the derivations. But in all sequences, the missiles count down one after the other, never simultaneously, and the fourth and last missile is always assumed to fire on the Nth day.

III. RESULTS

It will be seen in the derivations that N must be $\geqq R+3$. The probability $P(N)$ of firing the four missiles under the above launch model conditions is:

For $N = R + 3$

$$P(N) = P_{Ia}(N)\Big|_{N=R+3} = p^4$$

For $(R+4) \leqq N \leqq (2R+2)$

$$P(N) = P_{Ia}(N) + P_{IIIa}(N)$$

$$= \frac{p^3}{2}\frac{d^2}{dq^2}(q^{N-R-1}) - p^2 q \frac{d}{dq}(q^{N-R-2}) + 2pq^{N-R-1} + \frac{p^3 q}{2}\frac{d^2}{dq^2}(q^{N-R-2}) - p^2 q^2 \frac{d}{dq}(q^{N-R-3}) - (1-q^2)\,q^{2N-2R-4}$$

For $N = 2R + 3$

$$P(N) = [P_{Ib}(N) + P_{II}(N) + P_{IIIa}(N)]\Big|_{N=2R+3}$$

$$= q^R(R^2+2R+1) - q^{R+1}(3R^2+8R+4) + q^{R+2}(3R^2+10R+7) - q^{R+3}(R^2+4R+4) - q^{2R} + 4q^{2R+1} - 7q^{2R+2} + 4q^{2R+3}$$

For $N \geqq (2R+4)$

$$P(N) = P_{Ib}(N) + P_{II}(N) + P_{IIIb}(N) + P_{IV}(N)$$

$$= \frac{p^2}{q}[R - (R+1)q + q^{R+1}]\frac{d}{dq}(q^{N-R-1}) - \frac{p^3(R^2+R)}{2}q^{N-R-3} + 2pq^{N-R-2} + \frac{p^3(q^R - q^{2R})}{2}\frac{d^2}{dq^2}(q^{N-2R-1}) + \frac{p^3(1-q^R)}{2}\frac{d^2}{dq^2}(q^{N-R-1}) + p^2[-1 + (R+1)q^R - Rq^{R+1}]\frac{d}{dq}(q^{N-R-2}) + p(R^2-R-2)q^{N-2} - \frac{p(R^2+R)}{2}q^{N-3} + \frac{p(3R-R^2)}{2}q^{N-1} + \frac{p^4 q^{2R}}{6}\frac{d^3}{dq^3}(q^{N-2R-1})$$

The moment-generating function $M(\theta)$ is obtained as

$$
\begin{aligned}
M(\theta) = {} & p^4 e^{\theta(R+3)} \\
& + \frac{e^{\theta(R+4)}p}{(1-e^{\theta}q)}(2q^3 + 4p^2q - 3pq^2) \\
& + \frac{e^{\theta(R+5)}p^2}{(1-e^{\theta}q)^2}(5pq^2 - 2q^3) + \frac{e^{\theta(R+6)}}{(1-e^{\theta}q)^3}2p^3q^3 \\
& + \frac{e^{\theta(2R+3)}}{(1-e^{\theta}q)}p\left[-p^2q^R(R^2+2R+1)\right. \\
& \left. + p(2R+1)q^{R+1} - 2q^{R+2}\right] \\
& + \frac{e^{\theta(2R+4)}}{(1-e^{\theta}q)^2}p^2\left[-p(2R+3)q^{R+1} + 2q^{R+2}\right] \\
& - \frac{e^{\theta(2R+5)}}{(1-e^{\theta}q)^3}2p^3q^{R+2} - \frac{e^{\theta(R+4)}}{(1-e^{\theta}q^2)}(1-q^2)q^4 \\
& + \frac{e^{\theta(2R+3)}}{(1-e^{\theta}q^2)}(1-q^2)q^{2R+2} \\
& + e^{\theta(2R+3)}\left[q^R(R^2+2R+1)\right. \\
& - q^{R+1}(3R^2+8R+4) \\
& + q^{R+2}(3R^2+10R+7) - q^{R+3}(R^2+4R+4) \\
& \left. - q^{2R} + 4q^{2R+1} - 7q^{2R+2} + 4q^{2R+3}\right] \\
& + \frac{e^{\theta(2R+4)}}{(1-e^{\theta}q)}p\left[p^3q^{2R} - \frac{(R^2+R)}{2}q^{2R+1}\right. \\
& + \frac{(3R-R^2)}{2}q^{2R+3} + p(R^2+2R-2)q^{R+1} \\
& - p(R^2+4R+3)q^{R+2} + 2q^{R+2} \\
& - 3p^2q^{2R+1} + p^2(2R+6)q^{R+1} \\
& - p^2\frac{(R^2+5R+6)}{2}q^{2R+1} \\
& + p(R^2+3R+2)q^{2R+1} \\
& - p(R^2+R-3)q^{2R+2} \\
& \left. + (R^2-R-2)q^{2R+2}\right] \\
& + \frac{e^{\theta(2R+5)}}{(1-e^{\theta}q)^2}p^2\left[(R-1)q^{R+2} - (R+1)q^{R+3}\right. \\
& + p(R+6)q^{R+2} + 3p^2q^{2R+1} - p(R+6)q^{2R+2} \\
& \left. + (R+1)q^{2R+2} - (R-1)q^{2R+3}\right] \\
& + \frac{e^{\theta(2R+6)}}{(1-e^{\theta}q)^3}p^3\left[2q^{R+3} - 2q^{2R+3} + 3pq^{2R+2}\right] \\
& + \frac{e^{\theta(2R+7)}}{(1-e^{\theta}q)^4}p^4q^{2R+3}
\end{aligned}
$$

From here, the mean day μ (counted from the first day of the N-day period) on which one can expect to launch the fourth and last missile, is given by

$$
\begin{aligned}
\mu = \left.\frac{dM(\theta)}{d\theta}\right|_{\theta=0} = \Bigg\{ & p^4(R+3) \\
& + q\left(R+4+\frac{q}{p}\right)(2q^2+4p^2-3pq) \\
& + q^2\left(R+5+\frac{2q}{p}\right)(5p-2q) \\
& + \left(R+6+\frac{3q}{p}\right)2q^3 \\
& + q^R\left(2R+3+\frac{q}{p}\right)\left[-p^2(R^2+2R+1)\right. \\
& \left. + p(2R+1)q - 2q^2\right] \\
& + q^{R+1}\left(2R+4+\frac{2q}{p}\right)\left[-p(2R+3)+2q\right] \\
& - 2q^{R+2}\left(2R+5+\frac{3q}{p}\right) \\
& - q^4\left(R+4+\frac{q^2}{1-q^2}\right) + q^{2R+2}\left(2R+3+\frac{q^2}{1-q^2}\right) \\
& + q^R(2R+3)\left[(R^2+2R+1) + q^2(3R^2+10R+7)\right. \\
& - q(3R^2+8R+4) - q^3(R^2+4R+4) - 7q^{R+2} \\
& \left. + 4q^{R+3} - q^R + 4q^{R+1}\right] \\
& + q^R\left(2R+4+\frac{q}{p}\right)\left[p^3q^R - \frac{(R^2+R)}{2}q^{R+1}\right. \\
& + \frac{(R-R^2)}{2}q^{R+3} + p(R^2+2R-2)q \\
& - p(R^2+4R+3)q^2 + 2q^2 + Rq^{R+3} \\
& - 3p^2q^{R+1} + p^2(2R+6)q - p^2\frac{(R^2+5R+6)}{2}q^{R+1} \\
& + p(R^2+3R+2)q^{R+1} - p(R^2+R-3)q^{R+2} \\
& \left. + (R^2-R-2)q^{R+2}\right] \\
& + q^R\left(2R+5+\frac{2q}{p}\right)\left[(R-1)q^2 - (R+1)q^3\right. \\
& + 3p^2q^{R+1} + p(R+6)q^2 - p(R+6)q^{R+2}
\end{aligned}
$$

$$+ (R+1)\,q^{R+2} - (R-1)\,q^{R+3}]$$

$$+ q^{R}\left(2R + 6 + \frac{3q}{p}\right)(2q^{3} - 2q^{R+3} + 3pq^{R+2})$$

$$+ \left(2R + 7 + \frac{4q}{p}\right) q^{2R+3} \Big\} \text{ days}$$

Also from $M(\theta)$, the variance σ^2 of the $P(N)$ distribution would be given by

$$\sigma^2 = \nu_2 - \mu^2 = \frac{d^2M(\theta)}{d\theta^2}\bigg|_{\theta=0} - \mu^2$$

but this obviously would be quite complicated.

IV. DERIVATIONS

Let the four missiles be numbered arbitrarily but permanently as 1, 2, 3, and 4, and the pads as 1 and 2. When a given missile fires, the assumption of successive (nonsimultaneous) counting-down means that the next available missile starts counting, and this removes the randomness at each firing associated with the question of which missile fires next, and thus determines a specific order of firing. This means that any permutation of the firing order 1, 2, 3, 4 is logically equivalent to that order, as may be seen by a renumbering of the missiles. Thus, we need only calculate the results for this one firing order, 1, 2, 3, 4, since it is the only type of firing order that can possibly result.

But within this established firing order, 1, 2, 3, 4, four different sequences of events are possible, as will now be seen. Prior to the first day of the N-day period, missiles 1 and 2 have been erected ready to count on pads 1 and 2, respectively. The counting on missile 1 starts on the first day of the N-day period, and continues until missile 1 fires, no matter how many days this requires. On the day after missile 1 has fired (from pad 1), the R turnaround time on pad 1 commences (let us denote it by R_1 for reference purposes), in order to erect missile 3. But on the same day, the counting down starts on missile 2 on pad 2, and continues for as many days as are required to launch missile 2, simultaneously with the R_1 turnaround time proceeding on pad 1. However (see Diagram 1 and 2), missile 2 may fire within, or outside of, the R_1 turnaround period. If missile 2 fires within the R_1 period, missile 3 is still being erected on pad 1, and is not ready to start counting down until the day after the last day of R_1. On the other hand, if missile 2 fires on some given day after the end of the R_1 period, missile 3 is up on pad 1 and ready to count, but by assumption of no simultaneous countdowns missile 3 must wait until the day after missile 2 has been launched before it can start counting. Thus, we have two distinct possible paths for the sequence of events up to this point.

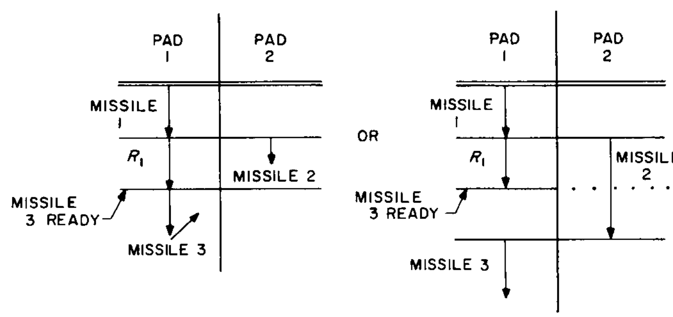

Diagram 1 and 2

But from here, whichever way missile 2 fires, on the day after it fires, the R turnaround time on pad 2 commences (let us denote it by R_2), in order to erect missile 4. The R_2 turnaround time is always proceeding simultaneously with the counting on missile 3 (on pad 1), for at least some part of the R_2 period (see Diagram 3 and 4). But here again the course of events may take one of two paths. If missile 3 (on pad 1) fires within R_2 (proceeding on pad 2), missile 4 is thus still in the process of being erected on pad 2, and is not ready to count until the day after the last day of R_2. On the other hand, if missile 3 (on pad 1) fires on some day after the end of the R_2 period, by assumption missile 4 must wait until missile 3 has fired before it can start counting. This is necessary even though missile 4 has been up and ready to count from the day after the last day of R_2.

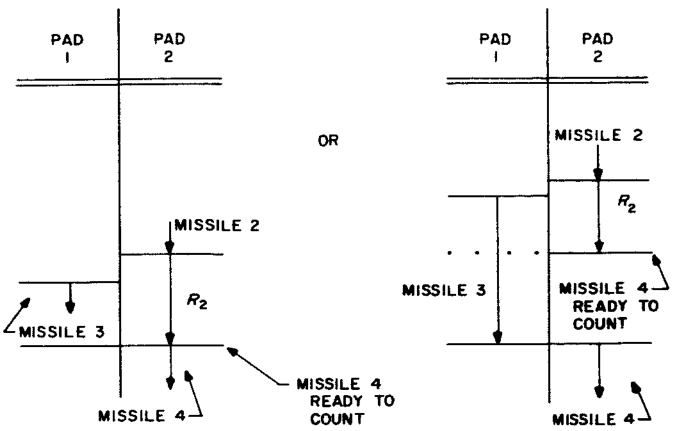

Diagram 3 and 4

Thus, one can see that there are four possible launch sequence cases, all with firing order 1, 2, 3, 4: case I—missile 2 fires within R_1, missile 3 fires within R_2; case II—missile 2 fires outside R_1, missile 3 fires within R_2; case III—missile 2 fires within R_1, missile 3 fires outside of R_2; case IV—missile 2 fires outside of R_1, missile 3 fires outside of R_2. The diagram for each case will be found below, which will clarify the above discussion. But whatever the case, missile 1 always starts counting on the first day of the N-day period, and missile 4 always is assumed to fire on the Nth day, whatever the positions of the launchings of missiles 2 and 3 within the N-day period.

We shall now look at each case separately.

A. Case I

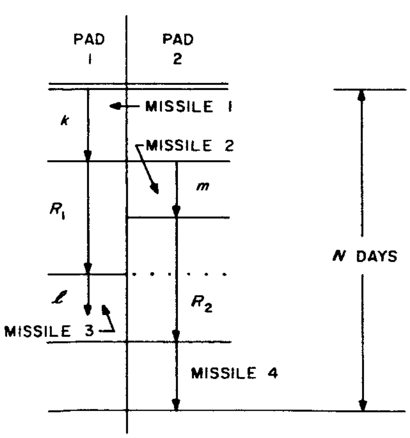

Diagram 5

The first case possible occurs when missile 2 fires within R_1, and missile 3 fires within R_2. Assume that missile 1 counts for k days, missile 2 for m days, and missile 3 for l days, as shown on Diagram 5 for this case. Missile 4 counts for the number of days remaining in N, or for $N - k - m - R$ days. The probability p_1 that missile 1 will fire on the kth day of its own counting is equal to the probability that it will fail to fire for $k - 1$ days, times the probability that it will fire in one day, or $p_1 = pq^{k-1}$. Similarly, $p_2 = pq^{m-1}$, $p_3 = pq^{l-1}$, and $p_4 = pq^{N-k-m-R-1}$.

The occurrence that missile 1 will take a particular k_0 days, with missile 2 taking a particular m_0 days, while missile 3 takes a particular l_0 days, with missile 4 then using $N - k_0 - m_0 - R$ days, is in itself a compound event whose probability equals the probability that the four subevents will happen together, the product $p_1p_2p_3p_4$. However, there can be different combinations of specific k's, m's, and l's within the overall N-day period. The compound firing event associated with a given k_0, m_0, l_0 set is an event mutually exclusive of the firing events associated with other k, m, l combinations. Thus, to arrive at a general probability expression $P_I(N)$ for getting a four-missile case I launch sequence, we must sum the product $p_1p_2p_3p_4$ over all allowable values of k, m, and l within the N-day period. Since this probability expression would give us the probability that we will get at least one of the possible launching k, l, m combinations, the expression would therefore also represent the probability of firing four missiles under case I conditions, with no simultaneous launches or countdowns, and with the fourth and last missile being fired on the Nth day.

$$P_I(N) = \sum_k \sum_m \sum_l p_1p_2p_3p_4$$

The smallest admissible value of N would be that for which all four missiles required only one day each to fire. We see from Diagram 5 that this value of N would be $N = R + 3$. In the entire launching model (not only here in case I), we have made no provision for the failure to launch all four missiles. We would therefore keep counting on each missile until all four were launched, which means that N has the upper limit infinity (here and in the other three cases).

It now remains for us to determine from the diagram the limits on k, m, and l for any given N-day period ($N \geqq R + 3$). The limits on m create a problem, however. The condition that missile 2 go off within $R_1 (= R)$ requires $m \leqq R$ always. But if we use the limit $1 \leqq m \leqq R$ in the triple sum above, we have not expressed in the setting-up of the triple sum that m is directly dependent on N for small values of N, in that for these values m must assume values between 1 and R_1 determined by the value of N. For the larger values of N, m is independent of N in that, although still $1 \leqq m \leqq R$, the value that m assumes is not dependent on N.

In order to provide for this, we must divide the range $R + 3 \leq N \leq \infty$ into two parts, one part being the subrange over which the upper limit on m is determined by N, and the other part being the subrange over which m can assume any value between 1 and R without depending on N. From the diagram, we see that for $N \leq 2R + 2$, $m \leq R$ without the necessity of stating it in the limit. Further, the subrange $R + 3 \leq N \leq 2R + 2$ is that range over which the upper limit on m depends on N. Thus, over this range, we will make the upper limit on m dependent on N, knowing that $m \leq R$ inherently, by our choice of upper limit on N. Over the other range $2R + 3 \leq N \leq \infty$, we will employ the limit $1 \leq m \leq R$ for m, but will arrange the other limits for k, l, and missile 4, such that they will be dependent on m, in order that the total number of days to fire all four missiles shall be N.

Case Ia. The philosophy of approach for the probability expression $P_{Ia}(N)$ valid over $R + 3 \leq N \leq 2R + 2$ is as follows (see Diagram 5). We shall make the days that missile 4 counts dependent on k and m, and l dependent on m also. Then m itself is made dependent on k, and out of a given N-day period, we consider k to be the independently varying quantity. The quantity k is then allowed to range over its maximum and minimum values, which takes care of the maximum and minimum values for m, l, and missile 4.

For k's maximum value out of any given N-day period, we see from the diagram that we must leave at least 1 day to shoot missile 4, and 1 day each to fire missiles 2 and 3 (or $R + 1$ days for both). Thus, $1 \leq k \leq N - R - 2$. Taking account of its dependence of k, the maximum value that m can assume is $N - k - R -$ (at least 1 day to shoot missile 4). Thus, $1 \leq m \leq N - k - R - 1$. In order to have missile 3 fire within R_2, $1 \leq l \leq m$. As stated, missile 4 counts for $N - k - m - R$ days. Thus, the probability $P_{Ia}(N)$ of launching the four missiles (under case I conditions), such that the last fires on the Nth day, for $R + 3 \leq N \leq 2R + 2$, is given by

$$P_{Ia}(N) = \sum_{k=1}^{N-R-2} \sum_{m=1}^{N-k-R-1} \sum_{l=1}^{m} p_1 p_2 p_3 p_4$$

$$= \sum_k \sum_m \sum_l pq^{k-1} pq^{m-1} pq^{l-1} pq^{N-k-m-R-1}$$

$$= p^4 q^{N-R-4} \sum_k \sum_m \sum_l q^l$$

or

$$P_{Ia}(N) = p^3 q^{N-R-3} \left[\frac{(N-R-1)(N-R-2)}{2} - \frac{q}{p}(N - R - 2) + \frac{q^2 - q^{N-R}}{p^2} \right]$$

$$R + 3 \leq N \leq 2R + 2$$

A firing sequence, such that the last missile goes off on a given Nth day, is mutually exclusive of firing sequences for other Nth days. Thus, the total probability P_{Iat} of getting at all a case I type of sequence over the range $R + 3 \leq N \leq 2R + 2$, is given by

$$P_{Iat} = \sum_{N=R+3}^{2R+2} P_{Ia}(N)$$

or

$$P_{Iat} = 1 - \frac{q}{1+q} - \frac{(R^2 + 3R + 2)}{2} q^R + (R^2 + 3R + 1) q^{R+1} - \frac{(R^2 + 3R + 2)}{2} q^{R+2} + \frac{q^{2R+3}}{1+q}$$

Case Ib. We shall now change the philosophy of approach for the range $2R + 3 \leq N \leq \infty$, and will consider m with the simple limits $1 \leq m \leq R$ to be the independently varying quantity, instead of k as in the $P_{Ia}(N)$ derivation. We will then make the other quantities dependent on m. Otherwise the philosophy and quantities are the same as above. From Diagram 5 we see still that in order to have missile 3 fire inside R_2, $1 \leq l \leq m$. But taking account of the dependence of k on m now, the maximum value that k can assume occurs if missile 4 requires only 1 day, and we must leave $R_2 (= R)$. Thus, $1 \leq k \leq N - m - R - 1$. The quantity p_4 still equals $pq^{N-k-m-R-1}$. Thus, the probability $P_{Ib}(N)$ of getting a case I type of launching sequence of four missiles, such that the last missile fires on the Nth day $(2R+3 \leq N \leq \infty)$ is given by

$$P_{Ib}(N) = \sum_{m=1}^{R} \sum_{l=1}^{m} \sum_{k=1}^{N-m-R-1} p_1 p_2 p_3 p_4$$

$$= p^4 q^{N-R-4} \sum_m \sum_l \sum_k q^l$$

or

$$P_{Ib}(N) = q^{N-R-3}\Big\{p^2(N-R-1)(R-Rq-q+q^{R+1}) - \frac{p^3(R^2+R)}{2} + p\,[q-(R+1)q^{R+1}+Rq^{R+2}]\Big\}$$

$$2R+3 \leq N \leq \infty$$

For the same reasons as before, the total probability P_{Ibt} of getting a case I type of firing sequence at all, for $2R+3 \leq N \leq \infty$, is

$$P_{Ibt} = \sum_{N=2R+3}^{\infty} P_{Ib}(N)$$

or

$$P_{Ibt} = \frac{(R^2+3R)}{2}q^R + (-R^2-3R-1)q^{R+1} + \frac{(R^2+3R+2)}{2}q^{R+2} + q^{2R+1} - q^{2R+2}$$

Finally, the total probability P_{It} of getting case I type of launching sequence at all is the sum of the two total probabilities for the two subranges of N, or

$$P_{It} = P_{Iat} + P_{Ibt}$$

$$P_{It} = 1 - \frac{q}{1+q} + q^{2R+1} - q^{2R+2} - q^R + \frac{q^{2R+3}}{1+q}$$

B. Case II

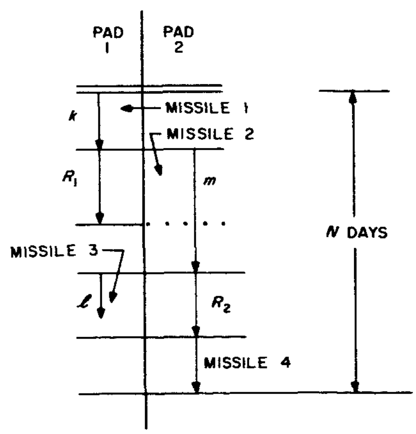

Diagram 6

The second case possible occurs when missile 2 fires outside of R_1 with missile 3 still firing within R_2. Again we shall assume missile 1 counts for k days, missile 2 for m days, missile 3 for l days. Under the conditions of this case $R+1 \leq m$, which implies from the diagram that the smallest admissible value for N is $N = 2R+3$. Further, we see that for $N \leq 2R+3$, we must always leave $R_2\,(=R)$, and that missile 3 will always fire within this R_2 period, independent of N. Thus, it is not necessary to do the case II calculations over two ranges of N, as in case I. For l, we need only use the limits $1 \leq l \leq R$.

Otherwise, the philosophy of approach is similar to case Ia. Out of any given N-day period, k is a maximum if m requires only its minimum $R+1$ days, and if missile 4 requires only 1 day. We must also leave $R_2\,(=R)$, no matter what day missile 3 fires on, as stated above. Thus, $1 \leq k \leq N-2R-2$. Making m dependent on k, the maximum value possible for m occurs when missile 4 requires only 1 day; we must also leave the $R_2\,(=R)$ period. Thus, $1 \leq m \leq N-k-R-1$. Finally, missile 4 counts for the days left in N after k, m, and $R_2\,(=R)$, or for $N-k-m-R$. Thus, $p_4 = pq^{N-k-m-R-1}$.

The probability $P_{II}(N)$ of getting a case II type of launching sequence of four missiles, such that the fourth missile fires on the Nth day, for $2R+3 \leq N \leq \infty$, is given by

$$P_{II}(N) = \sum_{k=1}^{N-2R-2}\ \sum_{m=R+1}^{N-k-R-1}\ \sum_{l=1}^{R} p_2 p_2 p_3 p_4$$

$$= p^4 q^{N-R-4} \sum_k \sum_m \sum_l q^l$$

or

$$P_{II}(N) = \frac{p^3(1-q^R)}{2}(N-2R-2)(N-2R-1)q^{N-R-3}$$

$$2R+3 \leq N \leq \infty$$

From this, the total probability P_{IIt} of getting a case II type of launching sequence at all is given by

$$P_{IIt} = \sum_{N=2R+3}^{\infty} P_{II}(N)$$

or

$$P_{IIt} = q^R - q^{2R}$$

C. Case III

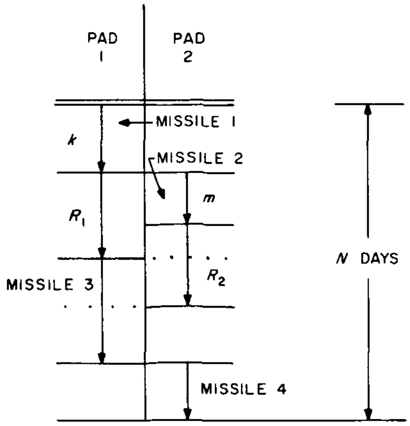

Diagram 7

The third case that can occur is where missile 2 fires within R_1, with missile 3 firing outside of R_2. The quantities are the same here as above, k, l, m.

Case IIIa. We are here again confronted with the same problem as in case I, in that we are constrained to have $m \leqq R$ always, but with the definite value of m dependent on N over one subrange, but not over the other. The value of m cannot exceed R if $N \leqq 2R + 3$ in this case. Within this range we will let k range over its maximum and minimum values, making m dependent on k and N. Out of any given N-day period, k is maximum when m equals its minimum 1, l equals its minimum 2 (2 in order that l will have missile 3 fire outside R_2), and missile 4 requires 1 day. Thus, $1 \leqq k \leqq N - R - 3$. Making l dependent on k, l has a maximum when missile 4 requires one day, and we must leave the $R_1 (= R)$ period. Thus, $2 \leqq l \leqq N - 1 - R - k$. In order to have missile 3 fire outside of R_2, we must require $1 \leqq m \leqq l - 1$ (see Diagram 7). The requirement $N \leqq 2R + 3$ then insures $m \leqq R$. The quantities p_1, p_2, p_3 are the same, but now missile 4 counts for $N - k - l - R$ days; thus, $p_4 = pq^{N-k-l-R-1}$. Lastly, we see from the above stated minimums for each quantity that the smallest admissible value for N is $N = R + 4$.

Thus, the probability $P_{IIIa}(N)$ of getting a case III type of launching sequence of four missiles, such that the last missile fires on the Nth day, for $R + 4 \leqq N \leqq 2R + 3$, is given by

$$P_{IIIa}(N) = \sum_{k=1}^{N-R-3} \sum_{l=2}^{N-k-R-1} \sum_{m=1}^{l-1} p_1 p_2 p_3 p_4$$

$$= p^4 q^{N-R-4} \sum_k \sum_l \sum_m q^m$$

or

$$P_{IIIa}(N) = q^{N-R-3} \left[\frac{(N-R-3)(N-R-2)p^3}{2} - p^2 q (N-R-3) + pq^2 - pq^{N-R-1} \right]$$

$$R + 4 \leqq N \leqq 2R + 3$$

For the same reasons as in other cases, the total probability P_{IIIat} of getting at all a case III type of launching sequence over $R + 4 \leqq N \leqq 2R + 3$, is

$$P_{IIIat} = \sum_{N=R+4}^{2R+3} P_{IIIa}(N)$$

or

$$P_{IIIat} = \frac{q}{1+q} - \frac{(R^2 + 3R + 2)}{2} q^{R+1} + (R^2 + 3R + 1) q^{R+2} - \frac{(R^2 + 3R + 2)}{2} q^{R+3} + \frac{q^{2R+4}}{1+q}$$

Case IIIb. For $N \geqq 2R + 4$, we shall again do as in case Ib, and consider m as the independently varying quantity, with simple limits $1 \leqq m \leqq R$. In order that missile 3 fire outside of R_2, the quantity l must be at least $m + 1$; l will be its maximum if k equals 1 day, and missile 4 requires only 1 day, and we must allow $R_1 (= R)$ also. Thus, $m + 1 \leqq l \leqq N - R - 2$. Making k dependent on l, k will be maximum if missile 4 requires only 1 day, and we must allow $R_1 (= R)$. Thus, $1 \leqq k \leqq N - R - 1 - l$. The quantity p_4 is the same as in case IIIa.

Thus, the probability $P_{IIIb}(N)$ of launching four missiles under case III conditions, such that the fourth fires on the Nth day, for $2R + 4 \leqq N \leqq \infty$, is

$$P_{IIIb}(N) = \sum_{m=1}^{R} \sum_{l=m+1}^{N-R-2} \sum_{k=1}^{N-R-1-l} p_1 p_2 p_3 p_4$$

$$= p^4 q^{N-R-4} \sum_m \sum_l \sum_k q^m$$

or

$$P_{IIIb}(N) = q^{N-R-3}\left\{\frac{p^3(1-q^R)(N-R-1)(N-R-2)}{2}\right.$$

$$- p^2(N-R-2)[1-(R+1)q^R + Rq^{R+1}]$$

$$+ p\left[q + (R^2-1)q^{R+1} - \frac{(R^2+R)}{2}q^R\right.$$

$$\left.\left. - \frac{(R^2-R)}{2}q^{R+2}\right]\right\} \qquad 2R+4 \leq N \leq \infty$$

Therefore, the total probability P_{IIIbt} of getting at all a case III type of launch sequence, for $2R+4 \leq N \leq \infty$, is

$$P_{IIIbt} = \sum_{N=2R+4}^{\infty} P_{IIIb}(N)$$

or

$$P_{IIIbt} = -(R^2+3R+1)q^{R+2}$$

$$+ \frac{(R^2+3R+2)}{2}q^{R+1} + \frac{(R^2+3R+2)}{2}q^{R+3}$$

$$+ q^{2R+2} - q^{2R+1} - q^{2R+3}$$

Finally, the total probability P_{IIIt} of getting at all a case III type of launching sequence of four missiles equals the sum of the total probabilities for the two subranges of N, or

$$P_{IIIt} = P_{IIIat} + P_{IIIbt}$$

$$P_{IIIt} = \frac{q}{1+q} + q^{2R+2} - q^{2R+1} - q^{2R+3} + \frac{q^{2R+4}}{1+q}$$

D. Case IV

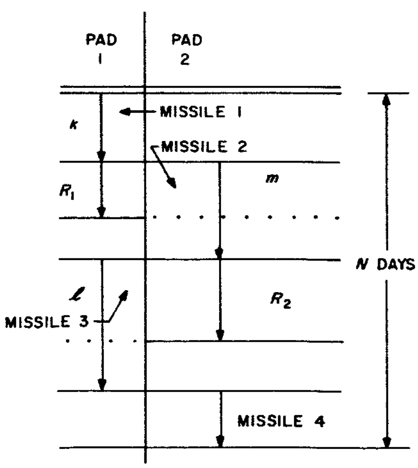

Diagram 8

The fourth and last possible case occurs when missile 2 fires outside of R_1, and missile 3 fires outside of R_2. Thus, from Diagram 8, the minimum limit for both m and l must be $R+1$, but the upper limit for them both is determined by the value of N. Thus, we do not have to calculate case IV in two parts.

Out of any given N-day period, k will be a maximum only if m and l are both $R+1$, and if missile 4 requires only 1 day to fire. Thus, $1 \leq k \leq N-2R-3$. Making m dependent on k, m will be a maximum only if l has its minimum value $R+1$, and if missile 4 requires only 1 day. Thus, $R+1 \leq m \leq N-k-R-2$. If we further make l dependent on k and m, l will be a maximum only if missile 4 requires 1 day. This implies $R+1 \leq l \leq N-k-m-1$. Missile 4 counts for $N-k-m-l$ days; thus, $p_4 = pq^{N-k-m-l-1}$.

From the minimums stated above, the smallest admissible value of N is $N = 2R+4$. Thus, the probability $P_{IV}(N)$ of launching four missiles under case IV conditions, such that the last fires on the Nth day, for $2R+4 \leq N \leq \infty$, is

$$P_{IV}(N) = \sum_{k=1}^{N-2R-3} \sum_{m=R+1}^{N-k-R-2} \sum_{l=R+1}^{N-k-m-1} p_1p_2p_3p_4$$

$$= \sum_k \sum_m \sum_l p^4 q^{N-R-4}$$

or

$$P_{IV}(N) = \frac{p^4q^{N-4}}{6}(N-2R-1)$$

$$\times (N-2R-2)(N-2R-3)$$

$$2R+4 \leq N \leq \infty$$

The total probability P_{IVt} of getting a case IV type of launching sequence at all is therefore given by

$$P_{IVt} = \sum_{N=2R+4}^{\infty} P_{IV}(N)$$

or

$$P_{IVt} = q^{2R}$$

The launching sequences (each associated with particular Nth days) of any one case are mutually exclusive of each other and also of all the launching sequences associated with particular Nth days for any other cases. Thus, we may add the probabilities for each case, over the ranges of N for which they are valid, to arrive at the general probability $P(N)$ of getting any possible type of launching sequence of four missiles, with no simultaneous countdowns or firings, such that the last missile fires on the specified Nth day. Thus,

For $N = R + 3$

$$P(N) = P_{Ia}(N)$$

For $R + 4 \leqq N \leqq 2R + 2$

$$P(N) = P_{Ia}(N) + P_{IIIa}(N)$$

For $N = 2R + 3$

$$P(N) = P_{Ib}(N) + P_{II}(N) + P_{IIIa}(N)$$

For $2R + 4 \leqq N \leqq \infty$

$$P(N) = P_{Ib}(N) + P_{II}(N) + P_{IIIb}(N) + P_{IV}(N)$$

These are the expressions given in Section II (Results).

Since we have made no provision for failure to launch all four missiles, we would keep counting until N = infinity. We must eventually achieve one of the four launch sequence case types above, and launch all four missiles. Thus, if we have included all possible launch sequences in the sample space that we have constructed for this problem, the total probability P_t over the whole sample space of launching all four missiles must be 1. But P_t must also equal the sum of the total probabilities of getting launch sequences of each case type, since these four cases are supposed to be the only ones in the sample space. We shall see if P_t is really equal to 1, as a check that we have included all possible cases.

$$P_t = P_{It} + P_{IIt} + P_{IIIt} + P_{IVt}$$

$$= \left(1 - \frac{q}{1+q} + q^{2R+1} - q^{2R+2} + \frac{q^{2R+3}}{1+q} - q^R\right)$$

$$+ (q^R - q^{2R}) + \left(\frac{q}{1+q} + q^{2R+2} - q^{2R+1} - q^{2R+3}\right.$$

$$\left. + \frac{q^{2R+4}}{1+q}\right) + (q^{2R}) = 1$$

We know, therefore, that we have included all possible launching sequences in the sample space. The total probability turning out to be 1 also implies that the algebraic calculations of the whole derivation must be correct also; and thus we have a second check on the derivation.

Lastly, $P(N)$ is in reality the probability function for a discrete random variable n, where n is the day on which the fourth and last missile fires. In other words, $P(N)$ is the probability that $n = N$. This means that the moment-generating function $M(\theta)$ can be calculated from the usual series

$$M(\theta) = E[e^{\theta N}] = \sum_{N=R+3}^{\infty} e^{\theta N} P(N)$$

with the result indicated in Section II (Results). As a check on the correctness of the $M(\theta)$ expression there, it is seen from the formula for $M(\theta)$ immediately above that for $\theta = 0$, $M(\theta)$ is then identical with the total probability P_t, and thus should equal 1. The reader will find this to be true for the $M(\theta)$ expression given in Section II.

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